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AN UPPER BOUND FOR THE NUMBER OF SPANNING TREES OF A GRAPH

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There is a well known formula for the number of spanning trees of a connected regular graph in terms of the eigenvalues of its adjacency matrix (see Biggs [1, p. 36]). Waller [3] has generalized this result and has given the corresponding formula for the number $S(G)$ of spanning trees of an arbitrary connected graph G. Application of the arithmetic mean, geometric mean inequality to his result yields an upper bound for this quantity:

$$S(G) \leqslant n^{-1}(2e/(n-1))^{n-1}, \tag{1}$$

where n and e are the numbers of vertices and edges of G. The purpose of this note is to demonstrate an alternative upper bound for $S(G)$.

Theorem. *Let G be a connected graph with n vertices, e edges and maximum valency d. Then*

$$S(G) \leqslant ((2e-d)/(n-1))^{n-1},$$

with equality if and only if G is the star graph on n vertices.

The bound of the theorem is not directly comparable with (1). It is an improvement if and only if

$$d/2e > 1 - n^{-(n-1)^{-1}}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $A = (a_{ij})$ be the adjacency matrix of G. The row sums of A are the valencies $d_1, d_2, \ldots, d_n$ of G and we may suppose without loss of generality that $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n = d$. It is well known (see Biggs [1, p. 34]) that all the cofactors of the matrix $M = (m_{ij})$ with entries

$$m_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ d_i & \text{if } i = j, \end{cases}$$

are equal, and their common value is the number of spanning trees of G. Note that M is nonnegative definite, because

$$M = \sum_{i<j} a_{ij} A_{ij},$$

where A_{ij} is the $n \times n$ matrix with 1 in the (i,i)th and (j,j)th entries, -1 in the (i,j)th and (j,i)th entries, and zeros elsewhere. Clearly the A_{ij} are nonnegative definite. [Alternatively, if $x = (x_1, x_2, \ldots, x_n)$, then

$$\begin{aligned} xMx^{\mathrm{T}} &= xDD^{\mathrm{T}}x^{\mathrm{T}} \\ &= (xD)(xD)^{\mathrm{T}} \geqslant 0, \end{aligned}$$

where D is the incidence matrix of G.] It follows that the submatrix N of M obtained by deleting the nth row and column is symmetric, nonsingular and nonnegative definite, and thus admits the representation

$$N = Q\Lambda Q^{-1} \tag{2}$$

for some invertible Q, where Λ is the diagonal matrix whose nonzero entries $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of N and are positive. Hence

$$\begin{aligned} S(G) &= \det N = \lambda_1 \lambda_2 \ldots \lambda_{n-1} \\ &\leqslant \left((n-1)^{-1} \sum_{i=1}^{n-1} \lambda_i \right)^{n-1} \end{aligned} \tag{3}$$

by the arithmetic mean, geometric mean inequality. By (2),

$$\sum_{i=1}^{n-1} \lambda_i = \operatorname{tr} \Lambda = \operatorname{tr} N = 2e - d,$$

where tr denotes trace, and the inequality of the theorem follows immediately. There is equality in (3) if and only if Λ is a multiple of the identity matrix, which implies by (2) that N is diagonal and that G is the star graph on n vertices.

Note. The problem of enumerating spanning trees is not only of interest to the graph theorist but has been shown by Mallion [2] to be related to the calculation of "ring currents" in molecular graphs.

References

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